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Some Remarks on Pathologies of Renormalization-Group Transformations for the Ising Model

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The results recently obtained by van Enter, Fernandez, and Sokal on non-Gibbsianness of the measure $\nu = T_b \mu_{\beta,h}$ arising from the application of a single decimation transformation T_b, with spacing b, to the Gibbs measure $\mu_{\beta,h}$ of the Ising model, for suitably chosen large inverse temperature β and nonzero external field h, are critically analyzed. In particular, we show that if, keeping fixed the same values of β, h, and b, one iterates a sufficiently large number of times n the transformation T_b, one obtains a new measure $\nu' = (T_b)^n \mu_{\beta,h}$ which is Gibbsian and moreover very weakly coupled.

KEY WORDS: Decimation; Renormalization-group; non-Gibbsianness; Finite-size conditions.

1. INTRODUCTION AND RESULTS

This note is motivated by a series of discussions with many colleagues and, in particular, with Giovanni Gallavotti and Joel Lebowitz, about the relationships between: (i) some recent results by van Enter, Fernandez, and Sokal (EFS) concerning non-Gibbsianness of some measures $T\nu$ obtained by applying a renormalization group transformation T to a Gibbsian measure ν,[(2)] and (ii) some recent results obtained by the present authors[(6-8)] on the application of finite-size conditions, of the form originally introduced in refs. 9 and 10, to the study of equilibrium and nonequilibrium properties of lattice spin systems near a first-order phase transition.

Joel Lebowitz suggested to us that we write a note to clarify these relationships by an example.

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We will consider a simple case: the Ising model in dimensions $d \geqslant 3$ at large inverse temperature β and nonzero external magnetic field h; we will denote by $\mu_{\beta,h}$ the associated unique Gibbs state.

In this case EFS prove that the *decimation transformation* T_b, on a scale b, gives rise, for suitable values of β and h, depending on b, to a non-Gibbsian distribution.

We prove here that, as an immediate consequence of the results in refs. 9, 10, and 6, if, for *exactly the same* thermodynamic parameters β and h we apply the decimation transformation $T_{b'}$ on any sufficiently large scale b', we obtain a measure which not only is Gibbsian, but is also weakly coupled (high temperature). In particular one can take $b' = b^n$ for all sufficiently large n; namely, one can iterate the EFS transformation to come back, in this way, to the set of Gibbs measures. Moreover, as a corollary, we obtain that $T_{b^n}\mu_{\beta,h}$ converges, at the level of the interaction, for n tending to infinity, to the trivial fixed point corresponding to a free system with the appropriate magnetization.

Let us now give some definitions.

The configuration space of the system is $\Omega = \{-1, 1\}^{\mathbf{Z}^d}$. The formal Hamiltonian is

$$H(\sigma) = -\tfrac{1}{2} \sum_{\langle i,j \rangle} \sigma_i \sigma_j - \tfrac{1}{2} h \sum_i \sigma_i \tag{1}$$

where $\langle i, j \rangle$ stands for a pair of nearest neighbors in $\mathbf{Z}^d$ and $h > 0$.

We use $\Omega_\Lambda = \{-1, 1\}^\Lambda$ to denote the configuration space in $\Lambda \subset \mathbf{Z}^d$.

Consider a *finite* region Λ in $\mathbf{Z}^d$ (in this case we write $\Lambda \subset\subset \mathbf{Z}^d$) and an arbitrary boundary condition τ outside Λ ($\tau \in \Omega_{\Lambda^c}$). The energy of a configuration σ in Λ is given by

$$H_\Lambda^\tau(\sigma) = -\tfrac{1}{2} \sum_{x,\, y \in \Lambda:\, |x-y| = 1} \sigma_x \sigma_y - \tfrac{1}{2} \sum_{x \in \Lambda} \Bigg(h + \sum_{y \notin \Lambda:\, |x-y| = 1} \tau_y \Bigg) \sigma_x \tag{2}$$

The finite-volume Gibbs measure in Λ, with τ boundary conditions, has the expression

$$\mu_\Lambda^\tau(\sigma) = \exp[-\beta H_\Lambda^\tau(\sigma)]/\text{normalization}$$

Notice that EFS use a different notation: they call magnetic field and denote by h our quantity βh.

The Dobrushin–Lanford–Ruelle (DLR) theory of Gibbs measures is based on the conditional probabilities π_Λ for the behavior of the system in a finite box $\Lambda \subset\subset \mathbf{Z}^d$ subject to a specific configuration in the complement of Λ. According to ref. 2, a probability measure whose conditional probabilities for finite subsets $\Lambda \subset\subset \mathbf{Z}^d$: $(\pi_\Lambda)_{\Lambda \subset\subset \mathbf{Z}^d}$ satisfy

$$\lim_{\Lambda' \uparrow \mathbf{Z}^d} \sup_{\omega_1, \omega_2 \in \Omega:\, (\omega_1)_{\Lambda'} = (\omega_2)_{\Lambda'}} |\pi_\Lambda f(\omega_1) - \pi_\Lambda f(\omega_2)| = 0 \tag{3}$$

(namely the conditional expectations in Λ of any cylindrical function f corresponding to different boundary conditions ω_1, ω_2, coinciding in $\Lambda' \supset \Lambda$, tend to coincide as Λ' tends to $\mathbf{Z}^d$) is called *quasilocal.*

A quasilocal probability measure on Ω satisfying also a so-called *non-nullity* condition, i.e., a sort of absence of hard-core exclusion, is called *Gibbsian* (see ref. 2 for more details).

In ref. 2 it is shown that the above notion of Gibbsianness of a measure is equivalent to the usual notion based on absolute summability properties of the interaction which gives sense to DLR equations.

The following theorem is proved in ref. 2 (see Theorem 4.7 therein).

Theorem 1. For each $d \geqslant 3$ and $b \geqslant 2$ there is a $\bar{\beta}$ and a function $\bar{h}(\beta)$ with $\bar{h}(\beta) > 0$ if $\beta > \bar{\beta}$ such that for all $\beta > \bar{\beta}$ and $h < \bar{h}$ the following is true: Let μ be a Gibbs measure for the d-dimensional Ising model described by the Hamiltonian (1) with inverse temperature β and magnetic field h. Then the renormalized measure $T_b \mu$, arising from a decimation transformation with spacing b, is not consistent with any quasilocal specification. In particular it is not the Gibbs measure for any uniformly convergent interaction.

We refer to Definitions 2.1–2.4 in ref. 2 for precise definitions concerning interactions.

Let us now state our result.

Theorem 2. For each $d \geqslant 3$ and $h > 0$ there is a b_0 and a β_0 such that for all $\beta > \beta_0$ and $b' > b_0$ the following is true: Let μ be the Gibbs measure for the d-dimensional Ising model described by the Hamiltonian (1) with inverse temperature β and magnetic field h. Then the renormalized measure $T_{b'} \mu$ arising from a decimation transformation with spacing b' is Gibbsian; moreover, the corresponding interaction is absolutely summable and the sum of all but the one-body terms tends to zero (in the norm $\mathscr{B}^1$ defined in ref. 2) as b' tends to infinity.

2. PROOF OF THEOREM 2

We will use definitions and notation of ref. 6, to which we refer for details.

Let us first recall the notion of finite-volume strong mixing condition (in its simplest form) introduced in ref. 6.

We say that the Gibbs measures μ_Λ^τ in Λ, with boundary condition τ outside Λ, satisfy the *strong mixing condition* in Λ, with parameters $C>0$, $\gamma>0$, and denote it by $SMC(\Lambda, C, \gamma)$, if, for all $x, y \in \Lambda$,

$$\sup_{\tau \in \Omega_{\Lambda^c}} |\mu_\Lambda^\tau(\sigma_x \sigma_y) - \mu_\Lambda^\tau(\sigma_x)\, \mu_\Lambda^\tau(\sigma_y)| \leqslant C \exp(-\gamma\, |x-y|) \tag{4}$$

In ref. 6 we have shown that, given C, γ, if $SMC(\cdot, C, \gamma)$ is satisfied for a sufficiently large cube $Q_L(C, \gamma)$ of side L, then there are $C'>0$, $\gamma'>0$ such that $SMC(\Lambda, C', \gamma')$ is satisfied for all arbitrarily large regions Λ which are *multiples* of the basic cube Q_L; where, given the odd integer L, a set Λ is said to be a multiple of the basic cube $Q_L(0)$ (of edge L centered at the origin)

$$Q_L(0) = \left\{ y \in \mathbf{Z}^d;\, |y_i| \leqslant \frac{L-1}{2},\, i = 1,..., d \right\}$$

if it is a union of translated cubes $Q_L(x) \equiv Q_L(0) + x$, $x \in \mathbf{Z}^d$, with disjoint interior,

$$\Lambda = \bigcup_{y \in Y} Q_L(Ly)$$

for some $Y \subset \mathbf{Z}^d$.

This property, namely the propagation to all larger scales of a finite-volume strong mixing condition, is called *effectiveness*.

Remark. Notice that in ref. 6 different notions of strong mixing were defined in a much more general setup. The possibility of using the particularly simple form given in (4) is a consequence of the peculiarities of the standard Ising model. In ref. 6 this condition was called $SMT(\Lambda, 1, C, \gamma)$.

It was shown in ref. 6 that the following proposition holds true:

Proposition 1. For all $d \geqslant 2$, $h>0$, there exists $L_0 = L_0(d, h)$ and $\beta_0 = \beta_0(d, h, L)$ such that $SMC(Q_L, C, \gamma)$ holds for all $L \geqslant L_0(d, h)$ provided $\beta > \beta_0(d, h, L)$.

Proof. Let us give here a proof of the above statement less sketchy than the one given in Section 5 of ref. 6.

Consider a cube $\Lambda = Q_L$ in $\mathbf{Z}^d$.

By FKG inequalities[3,4] and by taking the limit $\beta \to \infty$ of $\mu_\Lambda^{-\underline{1}}$, where $-\underline{1}$ is the configuration identically equal to -1, it follows that, if the ground-state configuration of $H_\Lambda^{-\underline{1}}(\sigma)$ with minus boundary conditions is identically equal to $+1$ for all $x \in \Lambda$, then the same holds for the ground-state configurations of $H_\Lambda^\tau(\sigma)$ with arbitrary boundary conditions τ.

We want now to prove that if $L > 2d/h$

$$\min_\sigma H_\Lambda^{-\underline{1}}(\sigma) = H_\Lambda^{-\underline{1}}(+\underline{1}); \qquad H_\Lambda^{-\underline{1}}(\sigma) > H_\Lambda^{-\underline{1}}(+\underline{1}) \quad \forall \sigma \neq +\underline{1} \tag{5}$$

namely that the configuration with all spins $+1$ in Λ is the unique ground state for $-\underline{1}$ boundary conditions.

Indeed, for every configuration $\sigma \in \Omega_\Lambda$ consider the union $C(\sigma)$ of all the closed unit cubes centered at each site $x \in \Lambda$: $\sigma_x = +1$. Consider, also, the union $D(\sigma)$ of the closed unit cubes centered at sites $x \in \mathbf{Z}^d$: $\sigma_x = -1$ (we recall that we set $\sigma_x = -1$, $\forall x \in \mathbf{Z}^d \setminus \Lambda$) and call $D^* = D^*(\sigma)$ the unique infinite connected component of $D(\sigma)$. The union $C(\sigma)$ splits into maximal connected components $C_1,..., C_k$. Among $C_1,..., C_k$ we select the subset $\bar{C}_1,..., \bar{C}_j$ of components touching D^*. We call them *outer components* and denote by $\gamma_1,..., \gamma_j$ their exterior boundaries (i.e., $\bar{\gamma}_i = \bar{C}_i \cap D^*$). We call $|\gamma_i|$ the measure of their boundaries γ_i and $|\theta(\gamma_i)|$ the measure (cardinality) of the interior $\theta(\gamma_i)$ of γ_i, namely the set of points that are separated from the boundary $\partial\Lambda$ by γ_i.

It is easy to prove the isoperimetric estimate

$$\sum_i |\theta(\gamma_i)| \leqslant \left(\sum_i \frac{|\gamma_i|}{2d} \right)^{d/(d-1)} \tag{6}$$

(see, for instance, Theorem 1.1 in ref. 11).

From (1.6) we get, for every $\sigma \in \Omega_\Lambda$,

$$\begin{aligned} H_\Lambda^{-\underline{1}}(\sigma) - H_\Lambda^{-\underline{1}}(-\underline{1}) &\geqslant -h \sum_i |\theta(\gamma_i)| + \sum_i |\gamma_i| \\ &\geqslant -h \sum_i |\theta(\gamma_i)| + 2d \left[\sum_i |\theta(\gamma_i)| \right]^{(d-1)/d} \end{aligned} \tag{7}$$

From (1.7) we get, for $L > 2d/h$,

$$H_\Lambda^{-\underline{1}}(\sigma) - H_\Lambda^{-\underline{1}}(-\underline{1}) \geqslant -hL^d + 2dL^{d-1} \equiv H_\Lambda^{-\underline{1}}(+\underline{1}) - H_\Lambda^{-\underline{1}}(-\underline{1}) \tag{8}$$

and the first equality in (5) is proven; the uniqueness of the minimum also follows from (7).

As we already said, from (5) we also get that $\forall \tau$, $+\underline{1}$ is the unique minimum for the energy.

Now, for every $L > 2d/h$, $C > 0$, $\gamma > 0$ given, if we choose a sufficiently large βh, it is easy to get the condition $SMC(Q_L, C, \gamma)$ [simply because

$\mu_\Lambda^\tau(\sigma_x = -1$ for some $x \in \Lambda) \to 0$ as $\beta \to \infty$, so that the Gibbs measure in Λ is, for every τ, a small perturbation of a δ-measure concentrated on the unique ground state $+\underline{1}$].

Now, from the effectiveness of $SMC(Q_L, C, \gamma)$ for L large enough, which has been proven in ref. 6, we are able to deduce properties of the renormalized interaction obtained by applying a block decimation transformation on a scale L. Before stating the result in Proposition 2 below, we need some more definitions.

Let $b = 2L$ and call $A(x)$ the cubic block $Q_L(bx)$ and $\alpha_x \in \Omega_{A(x)}$ the corresponding spin configuration. We call A the set of all the $A(x)$'s and we identify it with the subset of $\mathbf{Z}^d$ given by the union of the cubes $A(x)$.

For $\alpha \in \Omega_A$ let $H_A^{(r)}(\alpha)$ be the (formal) renormalized Hamiltonian obtained by integrating out the spins in $\mathbf{Z}^d \backslash A$. To be more precise, consider a big cube $\bar{\Lambda} \equiv Q_{\bar{L}}(0)$ centered at the origin with side $\bar{L} = (2p+1)L$, p integer. Choose free (empty) boundary conditions outside $\bar{\Lambda}$. For every $\sigma_{\bar{\Lambda}} \in \Omega_{\bar{\Lambda}}$ call

$$\alpha_{\bar{\Lambda}} = \sigma_{A \cap \bar{\Lambda}}; \qquad \eta_{\bar{\Lambda}} = \sigma_{\bar{\Lambda} \backslash A}$$

Let $H(\sigma_{\bar{\Lambda}}) \equiv H(\alpha_{\bar{\Lambda}}, \eta_{\bar{\Lambda}})$ be given by (1.1) and consider the renormalized Hamiltonian $H_A^{(r)}(\alpha_{\bar{\Lambda}})$ given by

$$\exp[-H_A^{(r)}(\alpha_{\bar{\Lambda}})] = \sum_{\eta_{\bar{\Lambda}}} \exp[-\beta H(\alpha_{\bar{\Lambda}}, \eta_{\bar{\Lambda}})]$$

This corresponds to applying to the Gibbs measure $\mu_{\bar{\Lambda}}$ a kind of decimation in $\bar{\Lambda} \backslash A$, that is, to construct the *relativization* of $\mu_{\bar{\Lambda}}$ to $\Omega_{\bar{\Lambda} \cap A}$. Call $\mu_{\bar{\Lambda}}^{(r)}$ the renormalized measure on $\Omega_{\bar{\Lambda} \cap A}$ obtained in this way:

$$\mu_{\bar{\Lambda}}^{(r)} = \frac{\exp[-H_A^{(r)}(\alpha_{\bar{\Lambda}})]}{Z_A}, \qquad Z_A = \sum_{\alpha_{\bar{\Lambda}}} \exp[-H_A^{(r)}(\alpha_{\bar{\Lambda}})] \equiv \sum_{\sigma_{\bar{\Lambda}}} \exp[-H(\sigma_{\bar{\Lambda}})]$$

One can repeat the same construction for any boundary condition $\tau \in \Omega_{\bar{\Lambda}^c}$ and get, in this way, the renormalized measure $\mu_{\bar{\Lambda}}^{(r),\tau}$.

Proposition 2. For L large enough we have:

(i) The result

$$\lim_{\bar{\Lambda} \uparrow \mathbf{Z}^d} \mu_{\bar{\Lambda}}^{(r),\tau} = \mu^{(r)}$$

holds independently of (the sequence of) boundary conditions τ.

(ii) $\mu^{(r)}$ is Gibbsian; there exists a corresponding interaction $(\Phi_V)_{V \subset A}$ (see ref. 2) which is absolutely summable.

(iii) The following condition is satisfied:

$$\sum_{V \ni A(0):\,|V|>1} \|\Phi_V\| = o(L)$$

Proof. Take any finite cube $\bar{\Lambda}$ with side $\bar{L} = (2p+1)L$ and τ boundary conditions outside $\bar{\Lambda}$. It is sufficient to notice that $\bar{\Lambda}\backslash A$ is a multiple of Q_L; so, by Proposition 1, for L sufficiently large the Gibbs measure $\mu^{\tau,\alpha}_{\bar{\Lambda}\backslash A}$ satisfies $SMC(\bar{\Lambda}, C', \gamma')$ for suitable C', γ', uniformly in $\bar{\Lambda}$. The same is true for any (not necessarily cubic) region $\bar{V}$ multiple of Q_L (see ref. 6 for more details).

Parts (i) and (ii) immediately follow from effectiveness. Indeed, let $\mu^{(r)}(\alpha_x \mid \alpha_y)$ be the conditional probability, with respect to the measure $\mu^{(r)}$ on Ω_A, of the configuration α_x in $A(x)$ given α_y in $A(y)$. Gibbsianness follows from nonnullity and quasilocality, which, in turn, follows from

$$|\mu^{(r)}(\alpha_x \mid \alpha_y) - \mu^{(r)}(\alpha_x)| \leqslant C'' \exp(-\gamma'' \,|x-y|) \tag{9}$$

for suitable C'', γ'', uniformly in $\bar{\Lambda}$, τ, α_x, α_y.

Condition (9) is a direct consequence of the strong mixing condition valid uniformly in $\bar{\Lambda}$ (effectiveness).

To get part (iii) we need more detailed estimates; it easily follows from the arguments developed in refs. 9 and 10, based on the cluster expansion, from Proposition 1 and Appendix 2 in ref. 6.

Let us now conclude the proof of Theorem 2. Let us use $\omega_x \in \{-1, +1\}$ to denote the value of the original spin variable σ_{bx} at the center bx of the cube $A(x)$. We set $\alpha_x = (\omega_x, \bar{\alpha}_x)$; $\bar{\alpha}_x \in \{-1, +1\}^{A(x)\backslash bx}$ is the restriction of α_x to $A(x)\backslash bx$.

Let $B = \{y = bx,\, x \in \mathbf{Z}^d\}$ be the sublattice of $\mathbf{Z}^d$ of spacing b. Consider the measure $\nu = T_b \mu$ obtained by applying the usual decimation transformation in $\mathbf{Z}^d\backslash B$ (relativization to Ω_B of the original Gibbs measure μ in $\Omega_{\mathbf{Z}^d}$).

We have

$$\nu(\omega_x \mid \omega_y) \equiv \mu(\omega_x \mid \omega_y) = \sum_{\bar{\alpha}_x} \mu^{(r)}(\bar{\alpha}_x, \omega_x \mid \omega_y) \equiv \sum_{\bar{\alpha}_x} \mu(\bar{\alpha}_x, \omega_x \mid \omega_y) \tag{10}$$

On the other hand,

$$\mu(\alpha_x \mid \omega_y) = \sum_{\bar{\alpha}_y} [\mu(\alpha_x \mid \bar{\alpha}_y, \omega_y) - \mu(\alpha_x \mid \bar{\alpha}^*_y, \omega^*_y)] \times \mu(\bar{\alpha}_y \mid \omega_y) + \mu(\alpha_x \mid \bar{\alpha}^*_y, \omega^*_y) \tag{11}$$

where α^*_y, ω^*_y denote a reference configuration (e.g., equal to all $+1$ in A_y).

From (9)–(11) we get the quasilocality condition (3); the nonnullity condition is trivially satisfied, so that we get the desired Gibbs property for ν. Absolute summability of the renormalized interaction immediately follows from the arguments of refs. 9 and 10 together with the estimate of the norm of the more-than-one-body interaction, which estimate vanishes as L increases to infinity. This concludes the proof of Theorem 2.

3. CONCLUSIONS

As it was noticed in ref. 2, the nonexistence of the renormalized interaction is a consequence of the presence of a first-order phase transition for the original model in $\mathbf{Z}^d \backslash B$ for particular values of $(\omega_x)_{x \in B}$ and suitable h and β; for example, $\omega_x = -1$, $\forall x$, and uniform positive h, exponentially in β near to the value $h^*(b)$, which is needed to compensate, in $\mathbf{Z}^d \backslash B$, the effect of the -1's in B and to give rise to a degeneracy in the ground state in $\mathbf{Z}^d \backslash B$ (see also ref. 5).

It seems clear, from the above analysis, that this pathology comes from the fact that, on a too short spatial scale b (with respect to the thermodynamic parameters and mainly the magnetic field h), the system is reminiscent of the existence of a phase transition for $h = 0$.

One needs to analyze the system on a large enough scale to put in evidence the uniqueness of the phase and the absence of long-range order. This scale, on which bulk effects become dominant with respect to surface effects, corresponds to the formation of a critical droplet of the stable phase; in other words, it is necessary to go to distances of this order to be sure that the boundary conditions have been screened out. The fact that on shorter distances the system is sensible to the boundary conditions and ordered is somehow related to the phenomenon of metastability taking place near a first-order phase transition.

The general philosophy suggested by the outcome of our Theorem 2 is that, when applying a renormalization group transformation, the system behaves as if it was weakly coupled, provided the scale of the transformation is chosen, depending on the thermodynamic parameters, in such a way that our strong mixing condition becomes effective; however, it is important to stress that the relevant length scale near a low-temperature coexistence line is not the correlation length of the unique pure phase, but, rather, the length of the critical droplet of the stable phase inside the metastable one.

Finally, we want to underline the fact that Theorem 2 is based on a finite-size condition related to a particularly simple shape: a cube.

As we discussed in ref. 6, an *effective* condition à la Dobrushin and Shlosman, implying their complete analyticity (see, for instance, ref. 1), could not be verified in the region of thermodynamic parameters that we are considering here. Indeed the Dobrushin–Shlosman finite-size condition involves the consideration of *arbitrary shapes*; it is clear that to exploit the presence of a positive magnetic field as a mechanism of screening we need, say, a plurirectangle with sufficiently large minimal edge. For not sufficiently "fat" and regular regions (for instance, pathological regions with many holes in the bulk), it is conceivable not only that a finite-size condition coming from the screening effect of h does not hold but, also, for special values of h and β, the Dobrushin–Shlosman complete analyticity can even fail. This is actually what EFS prove, in some cases, as a direct consequence of their methods to show non-Gibbsianness of some renormalized measures.

At the same time the equivalent of complete analyticity, not stated for *all* regions, but, rather, for arbitrarily large *but sufficiently regular* domains, directly follows from refs. 9 and 10 and the above-described finite-size condition on a suitable cube.

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